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Fuzzy Cubic Bézier Curve Approximation in Fuzzy Topological Digital Space

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ABSTRACT

By using Chang's fuzzy topology, a collection of digital data need to satisfy three conditions. Then, the collection will be known as fuzzy digital topology. The digital data in digital space consists of small dots or pixels and it is discrete. In this paper, first we consider the uncertainty in digital space by using fuzzy discrete set to produce fuzzy digital space and fuzzy point relation as the element in fuzzy digital space. Therefore, if fuzzy point relation satisfies the fuzzy digital topology conditions, then the fuzzy digital space will be known as fuzzy topological digital space. In fuzzy topological digital spaces, if two fuzzy points relation satisfy metric properties, then the space also can be known as fuzzy metric digital space. The main purpose in this paper is to build fuzzy cubic Bezier curve approximation in fuzzy topological digital spaces. Then by using fuzzy control point relation which satisfies the properties in fuzzy topological digital space, we show the visualization through some example .

Keywords: Fuzzy point relation, Fuzzy topological digital space, Fuzzy control point relation, Fuzzy cubic Bezier approximation.

1. Introduction

Digital space means some output that will be displayed on a screen of a digital device such as laptop, computer, tablet or smart phone Axel (2000). A digital either in signal or data form in electronics technology will generates, stores and processing data having two possible states either positive (denoted as 1) and non positive (denoted as 0). These two digits are called binary digits (BIT) (refer Warford (2005),Godse and A (2009)). This concept of digital has been spread wisely in many field and one of them is digital geometry (refer Kiselmann (2004),Melin (2008)). Digital geometry means geometry of computer images. Melin (2008) has studied continuous functions between digital spaces and also curves that can be generated by continuous function in digital spaces. The computer graphic deal with the image is consists of small dots, pixels and it is discrete Solomon (2006). First objective of this paper is to introduce fuzzy digital space. If the universal set in digital space is discrete, then by using fuzzy discerete set theory and some properties we can introduce fuzzy digital space. Fuzzy discrete set Kaspshitskaya et al. (1990) has been extended from fuzzy set theory from Zadeh (1965). Fuzzy discrete set has been used in image processing by Rosenfeld (1979) and Rosenfeld (1984). In fuzzy metric space field, there are two approaches either by using fuzzy number to define metric in ordinary space (refer Kaleva and Seikkala (1984),George and Veeramani (1997),Gregori and Romaguera (2000)) or by using real number to measure the distance between fuzzy sets (refer Diamond and Kloeden (1990),Chaudhur and Rosenfeld (1996),Voxman (1998)). In image processing, a concept of topological digital space concept has been introduced by Herman (1998). From that, Kopperman (1998) introduced Alexandroff's topology and Melin (2008) generalize Khalimsky's topology. Then, Muhammad and Tahir (2013) consider a fuzziness perspective in topological digital space and introduce fuzzy topological digital space (FTDS). In fuzzy topology field, Chang's fuzzy topology is a collection of fuzzy subset in universal set that must be fulfilled three conditions (see Chang (1968)). Many researchers began to produce many concepts by using Chang's fuzzy topology such as openness, closedness, continuous, neighbourhood, interior and closure (see Chang (1968),Pu and Liu (1980),Wong (1974),Waren (1974)). Then Shah and Wahab (2017) using Chang's fuzzy topology to focus more on characteristic of fuzzy topology in fuzzy digital space and then introduced FTDS where the collection of digital data satisfies three conditions. In this paper, we are going to use fuzzy point relation as an element in fuzzy digital space and satisfies some properties.

Curve and surface are the main parts to design a geometric modeling where CAGD (Computer Aided Geometric Design) is a system of design process. These geometric models that have been form will help the user to make analysis, decision and conclusion. There are two most important terms in the field of curve and surface design which is interpolation and approximation (see Solomon (2006)). In this paper, we focus on to build fuzzy Bezier curve approximation. In collecting data process, there will always have some major problems that can be found in specific tool or by some instrument. The data always imprecise or error and also vague. The uncertainty data problem that occurred is solving by using fuzzy set theory Zadeh (1965) and fuzzy number (refer Klir et al. (1997)). In Wahab et al. (2009) has introduced fuzzy control point to blend with Bernstein basis function to build fuzzy Bezier Curve. Then, these new findings have been extended and spread wisely by many author (refer Wahab et al. (2009),Rozaimi et al. (2011),Wahab and Zulkifly (2017)). In Wahab and Zulkifly (2017) has been introduced fuzzy control point relation (FCPR) to generate a new fuzzy Bezier curve modeling. This will become our main method in building fuzzy Bezier curve in FTDS. Digital space is a product of two discrete universal sets. The first section in this paper is to build fuzzy digital space where the digital space with membership function $[0, 1]$ using fuzzy discrete set theory and fuzzy point relation as an element in fuzzy digital space. Meanwhile in the next section if the collection of fuzzy point relation satisfies Chang's fuzzy topology, therefore the space now will be known as FTDS. For every fuzzy point relation in fuzzy digital space, there exist a distance between them. Therefore if the distance satisfies some conditions of metric theory, therefore the space can be known as fuzzy metric digital space. This will be discussed in the third section. This paper is cubic case where n=3, then we show the visualization of fuzzy cubic Bezier curve approximation in FTDS by using FCPR. The result will show at the end of this paper.

2. Preliminaries

In this section, consider the universal set in digital space is discrete. Before we build fuzzy digital space, we discuss some theory that is needed such as fuzzy discrete set, fuzzy point and fuzzy point relation. Before we go through the fuzzy discrete set concepts, let's discuss the meaning of digital space.

2.1 Digital Space

Suppose $X = \{x_1, x_2, x_3, ..., x_n\}$ be a discrete universal set and $A \subset X$ be a collection of any discrete element (pixels) in X.Therefore we can conclude that the collection of these pixels is called digital data and each element of these pixels called digital points.In Warford (2005) mention that the words digital means the signal from memory and can only have a fixed number of values. All computer system operates with the binary numbers which use only two digits either 0 or 1. Binary number or BiT is a unit of information expressed. In binary language 0 means OFF and 1 means ON. There are no specific definition. Therefore, discrete set of $A \subset X$ can be characterize by some function, χ_A such that $\chi_A : A \subset X \to \{0, 1\}$. For digital space definition, let's defined by using product of Cartesian (see Leung and Chen (1992)) between two discrete-universal set.

Definition 2.1. *Let X and Y be two discrete-universal sets. Digital space is the product of Cartesian of X and Y that can be denoted as $X \times Y$ and also can be called as 2D digital space where the collection of digital data (must in ordered pair) in 2D digital space is characterized by {0,1}*

$$X \times Y \times \{0, 1\} = \{((x_i, y_i), \chi_{X\times Y}(x_i, y_i))\} \tag{1}$$

such that $\{x_i\}_{i=1}^n \in X$ and $\{y_i\}_{i=1}^n \in Y$. The symbol of χ_A is some characteristic function and $\chi_{X\times Y}(x_i, y_i) \in \{0, 1\}$.

2.2 Fuzzy Discrete sets

Consider the digital data is discrete. Fuzzy discrete set has been introduced by Kaspshitskaya et al. (1990) and also has been mentioned by Voxman (2001).

Definition 2.2. *Kaspshitskaya et al. (1990) Let discrete universal set $X = \{x_1, x_2, x_3, ..., x_n\}$ and $A \subset X$ as a collection of any discrete element in X. Then $\tilde{A}$ is said to be fuzzy discrete set in $X \times I$ where the membership function $\mu_A : A \subset X \to I$ where $I = [0, 1]$ (see figure1). Fuzzy discrete set $\tilde{A}$ also can be defined as $\tilde{A} = \{(x_i, \mu_A(x_i)) : \{x_i\}_{i=1}^n \in X, \mu_A(x_i) \in I\}$*

Definition 2.3. *Kaspshitskaya et al. (1990) If $\tilde{B}$ is fuzzy discrete subset in $\tilde{A}$ such that $\tilde{B} \subset \tilde{A}$, then $\mu_B(x_i) \leq \mu_A(x_i), \forall \{x_i\}_{i=1}^n \in X$. Fuzzy discrete subset is shown in figure 1*

Definition 2.4. *Kaspshitskaya et al. (1990) Let X discrete-universal set and $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{E} \subset X \times I$ are fuzzy discrete set where $I = [0, 1]$. Then*
i. $\tilde{A} = \tilde{B} \Leftrightarrow \mu_A(x_i) = \mu_B(x_i)$
ii. $\tilde{A} \subset \tilde{B} \Leftrightarrow \mu_A(x_i) \leq \mu_B(x_i)$
iii. $\tilde{C} = \tilde{A} \cup \tilde{B} \Leftrightarrow \mu_C(x_i) = max\{\mu_A(x_i), \mu_B(x_i)\}$
iv. $\tilde{D} = \tilde{A} \cap \tilde{B} \Leftrightarrow \mu_D(x_i) = min\{\mu_A(x_i), \mu_B(x_i)\}$
v. $\tilde{E} = \tilde{A}' \Leftrightarrow \mu_E(x_i) = min\{\mu_A(x_i), 1 - \mu_A(x_i)\}$
where $\forall \{x_i\}_{i=1}^n \in X$ (see figure 2).

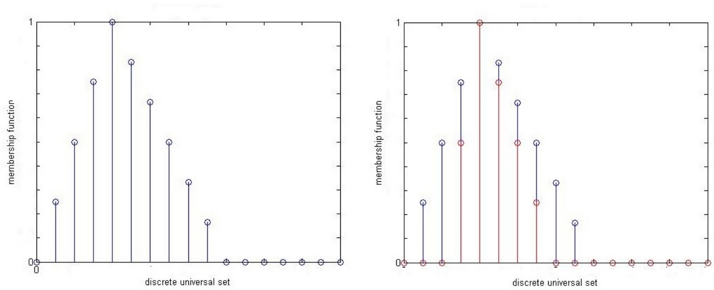

Figure 1: Left:Fuzzy Discrete Set; Right:Fuzzy Discrete Subset

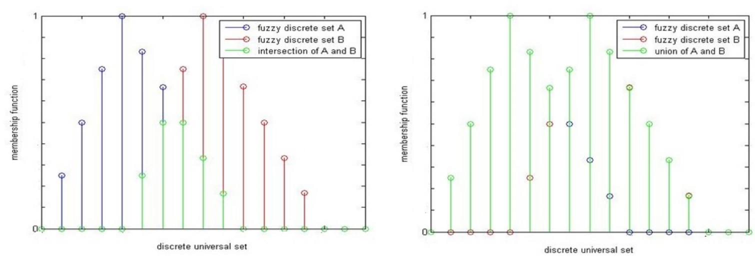

Figure 2: Left:Intersection of two fuzzy discrete set; Right:Union of two fuzzy discrete set

Definition 2.5. *Kaspshitskaya et al. (1990) If $\tilde{A}_i$ collection of fuzzy discrete set in discrete-universal set,$X \times I$ where $[0, 1]$,then $\tilde{C} = \bigcup_{i=1}^{n} \tilde{A}_i$ and $\tilde{D} = \bigcap_{i=1}^{n} \tilde{A}_i$ defined by $\mu_C(x_i) = \sup_{i=I}\{\mu_{A_i}(x_i)\}$ and $\mu_D(x_i) = \inf_{i=I}\{\mu_{A_i}(x_i)\}$ where $\{x_i\}_{i=1}^{n} \in X$.*

2.3 Fuzzy Point

Pu and Liu (1980) has introduced fuzzy point theory as a single fuzzy element. Then Wong (1974) and Waren (1974) started to expand this theory to other fields. Consider the universal set is discrete.

Definition 2.6. *Wong (1974) Let $X \times I$ be a collection of points in $1D$ where $[0, 1]$. Fuzzy point $\tilde{P}$ in $X \times I$ is fuzzy discrete set with membership function*

$$\mu_P(x) = \begin{cases} 0 < y_i < 1 & \text{if } x = x_i \\ 0 & \text{if } x \neq x_i \end{cases} \tag{2}$$

Fuzzy point, $\tilde{P}$ have support x_i with $x_i = \{x_1, x_2, x_3, ..., x_n\}$ or $\{x_i\}_{i=1}^n$ correspond to value y_i where $y_i = \{y_1, y_2, y_3, ..., y_n\}$. Fuzzy point can be written as $\tilde{P} = \{\langle x_i, y_i\rangle : y_i \in (0,1)\}$.

Definition 2.7. *Wong (1974) Let $\tilde{P}$ be a fuzzy point and $\tilde{A}$ is fuzzy discrete set in $X \times I$ where $[0,1]$. Then $\tilde{P}$ is an element for fuzzy discrete set $\tilde{A}$ denoted by $\tilde{P} \in \tilde{A} \subset X \times I$ if and only if $\mu_P(x_i) \leq \mu_A(x_i)$ for all $x_i \in X$. Therefore, every fuzzy discrete set $\tilde{A}$ is the union of all fuzzy point which belongs to $\tilde{A}$.*

2.4 Fuzzy Point Relation

Fuzzy point relation has been produced by using the fuzzy relation (see Zadeh (1965)) by Wahab and Zulkifly (2017). In Wahab and Zulkifly (2017) introduced this theory to be used as FCPR to conclude the shape of spline curve or surface. Consider the space of point is an element in fuzzy discrete set.

Definition 2.8. *Wahab and Zulkifly (2017) Let A and B be a collection of point with non-empty sets and $A, B \subseteq \mathbb{R}$, then fuzzy point relation is defined as*

$$\tilde{S} = \{(P_i, z_i) : z_i \in (0,1)\} \tag{3}$$

where $P_i = (x_i, y_i) \in A \times B, \forall\{x_i\}_{i=1}^n \in X, \forall\{y_i\}_{i=1}^n \in Y$, and the membership function $z_i : A \times B \subset X \times Y \to (0,1)$

Definition 2.9. *Wahab and Zulkifly (2017) Let $A, B \subseteq \mathbb{R}$ with $\tilde{M} = \{(x_i, \alpha_i) : \alpha \in (0,1)\}$ and $\tilde{N} = \{(y_i, \beta_i) : \beta_i \in (0,1)\}$ represent two fuzzy points. Then Equation 3 is a fuzzy point relation on $\tilde{M}$ and $\tilde{N}$ if $z_i \leq \alpha_i$ and $z_i \leq \beta_i$ for every $(x_i, y_i) \in A \times B$*

Definition 2.10. *Wahab and Zulkifly (2017)Let $\tilde{S}$ and $\tilde{T}$ be fuzzy point relation in the collection of points with $A, B \subseteq \mathbb{R}$, then the union and intersection of $\tilde{S}$ and $\tilde{T}$ are defined by*
i. $\tilde{S} \cup \tilde{T} = \max\{z_i, w_i\}$ where $z_i \cup w_i \in (0,1)$
ii. $\tilde{S} \cap \tilde{T} = \min\{z_i, w_i\}$ where $z_i \cap w_i \in (0,1)$

3. Fuzzy Digital Space

Digital space has been introduced in Equation 1. By using fuzzy discrete set, the fuzzy digital space can be defined as follows.

Definition 3.1. *Let $X \times I$ and $Y \times I$ be two discrete-universal sets with $[0,1]$. Fuzzy digital space is the product of Cartesian $X \times I$ and $Y \times I$ and can be* *denoted as* $X \times Y \times I$ *where* $I = I \times I$*. This* $X \times Y \times I = \{((x_i, y_i), \mu_{X\times Y}(x_i, y_i))\}$ *such that* $\{x_i\}_{i=1}^{n} \in X, \{y_i\}_{i=1}^{n} \in Y$ *and* $\mu_{X\times Y\times I}((x_i, y_i) \in I$ *is membership function of* $X \times Y \times I$ *and* $I = I \times I$ *and also can be known as 2D fuzzy digital space.*

Definition 3.2. *Let* A *and* B *be a collection of point with non-empty sets and* $A, B \subseteq \mathbb{R}$*. If* $\tilde{A} \subseteq X \times I$ *and* $\tilde{B} \subseteq Y \times I$ *then there exist fuzzy point relation* $\tilde{S} \subseteq X \times Y \times I$*.*

Definition 3.3. *Let* $\tilde{S}_i$ *collection of fuzzy point relation in* $X \times Y \times I$*, then*
i. $\bigcup_i \tilde{S}_i = \sup_i \{z_i\}$
ii. $\bigcap_i \tilde{S}_i = \inf_i \{z_i\}$

Figure 3 below show $\tilde{S} \subseteq X \times Y \times I$

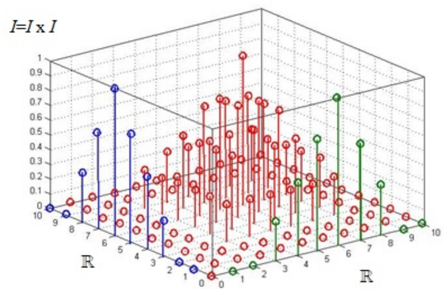

Figure 3: Fuzzy point relation in fuzzy digital space

4. Fuzzy Topological Digital Space

The concept of FTDS has been introduced in Shah and Wahab (2017) where the collection of digital data need to satisfy three conditions. In this paper, we want to improve the previous definition of FTDS in Kiselmann (2004) by show step by step to generate this concept by produce a fuzzy digital space concept first in previous section and fuzzy point relation as an element in fuzzy digital space. Let $X \times Y \times I$ as fuzzy digital.

Definition 4.1. *Suppose* $X \times Y \times I \neq \emptyset$ *and* $\tilde{\tau}$ *is a collection of fuzzy point relation on* $X \times Y \times I$ *if* $\tilde{\tau}$ *satisfied three conditions:*

i. $\emptyset, X \times Y \times I \in \tilde{\tau}$
ii. If $\tilde{S}, \tilde{T} \in \tilde{\tau}$, *then* $\tilde{S} \cap \tilde{T} \in \tilde{\tau}$
iii. If $\tilde{S}_i \in \tilde{\tau}$, *then* $\bigcup_{i \in I} \{\tilde{S}_i\} \in \tilde{\tau}$
Then $\tilde{\tau}$ *is fuzzy digital topology on* $X \times Y \times I$. *The pair* $(X \times Y \times I, \tilde{\tau})$ *will be known as fuzzy topological digital space (FTDS).*

Example 4.1. *Let* $\{(P_1, z_1), (P_2, z_2), (P_3, z_3)\} \in \tilde{S}$ *and* $\tilde{S}$ *be a fuzzy point relation of* $X \times Y \times I$ *with the membership function of* $z_1 = 0.3, z_2 = 0.5$ *and* $z_3 = 0.4$. *Take* $(P_i, z_i) \in \tilde{S}$ *as an example. Then,* $\tilde{\tau} = \{\emptyset, \tilde{S}, X \times Y \times I\}$ *is a fuzzy digital topology with the membership function of* $\emptyset$ *and* $X \times Y \times I$ *are* $z_4 = 0$ *and* $z_5 = 1$ *where satisfying three conditions:*
i. $\emptyset, X \times Y \times I \in \tilde{\tau}$.
ii. If $\emptyset, \tilde{S} \in \tilde{\tau}$, *then* $\emptyset \cap \tilde{S} = \min\{z_4, z_1\} = 0 = z_4 = \emptyset \in \tilde{\tau}$,
If $\emptyset, X \times Y \times I \in \tilde{\tau}$, *then* $\emptyset \cap \{X \times Y \times I\} = \min\{z_4, z_5\} = 0 = z_4 = \emptyset \in \tilde{\tau}$,
If $\tilde{S}, X \times Y \times I \in \tilde{\tau}$, *then* $\tilde{S} \cap \{X \times Y \times I\} = \min\{z_1, z_5\} = 0.3 = z_1 = \emptyset \in \tilde{\tau}$.
iii. If $\emptyset, \tilde{S} \in \tilde{\tau}$, *then* $\emptyset \cup \tilde{S} = \max\{z_4, z_1\} = 0.3 = z_1 = \tilde{S} \in \tilde{\tau}$,
If $\emptyset, X \times Y \times I \in \tilde{\tau}$, *then* $\emptyset \cup \{X \times Y \times I\} = \max\{z_4, z_5\} = 1 = z_5 = X \times Y \times I \in \tilde{\tau}$,
If $\tilde{S}, X \times Y \times I \in \tilde{\tau}$, *then* $\tilde{S}\{X \times Y \times I\} = \max\{z_1, z_5\} = 1 = z_5 = X \times Y \times I \in \tilde{\tau}$
$\emptyset \cup \tilde{S} \cup \{X \times Y \times I\} = \sup\{z_4, z_1, z_5\} = 1 = z_5 = X \times Y \times I \in \tilde{\tau}$.

Definition 4.2. *Let* $\tilde{\tau_1}$ *and* $\tilde{\tau_2}$ *fuzzy digital topology on* $X \times Y \times I$. *If the inclusion relation* $\tilde{\tau_1} \subset \tilde{\tau_2}$ *holds, then* $\tilde{\tau_2}$ *is finer than* $\tilde{\tau_1}$ *and* $\tilde{\tau_1}$ *is coarser than* $\tilde{\tau_2}$.

5. Fuzzy Metric Digital Space

In this section, we would like to measure the distance between every fuzzy point relation in fuzzy digital space by using condition from fuzzy metric concept. Then, fuzzy metric in fuzzy digital space can be defined as follows.

Definition 5.1. *Let* $X \times Y \neq \emptyset$ *and* $d : X \times Y \times I \to [0, \infty)$ *is a mapping where* $I = [0, 1]$.*Then* $(X \times Y \times I, d)$ *is said to be a fuzzy metric digital space if for any* $\{(P_1, z_1), (P_2, z_2), (P_3, z_3)\} \in \tilde{S}$ *where* $\{z_i\}_{i=1}^{3} \in (0, 1)$ *and* $\tilde{S}$ *in* $X \times Y \times I$, *d satisfies the following three conditions,*
i. $d((P_1, z_1), (P_2, z_2)) = 0$ *if and only if* $P_1 = P_2$ *and* $z_1 = z_2 = 1$*(nonnegative)*
ii. $d((P_1, z_1), (P_2, z_2)) = d((P_2, z_2), (P_1, z_1))$*(symmetric)*
iii. $d((P_1, z_1), (P_2, z_2)) + d((P_2, z_2), (P_3, z_3)) \geq d((P_1, z_1), (P_3, z_3))$ *(triangle inequality).*
Then, d is known as a fuzzy digital metric defined in $X \times Y \times I$ *and* $d((P_1, z_1), (P_2, z_2))$ *is called fuzzy digital distance between two elements in fuzzy point relation.*

Note that fuzzy digital metric space have $\{\tilde{P}_i\}_{i=1}^n$ as their element where $\{\tilde{P}_i\}_{i=1}^n$ actually is $\{\tilde{P}_i\}_{i=1}^n \in \tilde{S}$ in fuzzy point relation. We show some example as below.

Example 5.1. *Suppose $X \times Y \times I$ is fuzzy digital space. The distance between $(P_1, z_1), (P_2, z_2)$ in $\tilde{S}$ is defined by $d((P_1, z_1), (P_2, z_2)) = (d((P_1, P_2), \min\{z_1, z_2\})$ where $d((P_1, P_2)$ is the distance between P_1 and P_2 defined $X \times Y \times I$. Then $(X \times Y \times I, d)$ is a fuzzy metric digital space where satisfied the properties below.*
i. Suppose$(P_1, z_1)$and(P_2, z_2) in $\tilde{S}$.It is obvious that $d((P_1, z_1), (P_2, z_2)) = 0$ if and only if $d((P_1, P_2)$ and $\min\{z_1, z_2\} = 1$ which is equal to that $P_1 = P_2$ and $z_1 = z_2 = 1$.
ii. For any $\{((P_1, z_1), (P_2, z_2))\} \in \tilde{S}$, then $d((P_1, z_1), (P_2, z_2)) = d((P_1, P_2), \min\{z_1, z_2\}) = d((P_2, P_1), \min\{z_2, z_1\}) = d((P_2, z_2), (P_1, z_1))$
iii. For any$\{((P_1, z_1), (P_2, z_2)), (P_3, z_3)\} \in \tilde{S}$, then $d((P_1, z_1), (P_3, z_3)) = (d((P_1, P_3), \min\{z_1, z_3\}) \leq (d(P_1, P_2) + d(P_2, P_3), \min\{z_1, z_2, z_3\}) = (d(P_1, P_2), \min\{z_1, z_2\} + (d(P_2, P_3), \min\{z_2, z_3\}) = d((P_1, z_1), (P_2, z_2)) + d((P_2, z_2), (P_3, z_3))$

Definition 5.2. *Let $(X \times Y \times I, d)$ is a fuzzy metric digital space. Then $\tilde{P}_i$ is called fuzzy open if and only if for every $\tilde{P}_i \in \tilde{S}$, there exist $r > 0$ such that $S(\tilde{P}_i, r) \leq \tilde{S}$ where $S(\tilde{P}_i, r)$ means sphere with the center $\tilde{P}_i$ and radius r.*

Below we show the link between fuzzy metric digital spaces with fuzzy digital topology's conditions.

Theorem 5.1. *Let $(X \times Y \times I, d)$ be a fuzzy metric digital space and $\tilde{\tau}$ denote the set of all fuzzy open in $X \times Y \times I$. Then $\tilde{\tau}$ has the following properties:*
i. $\emptyset, X \times Y \times I \in \tilde{\tau}$
ii. Arbitrary union of members of $\tilde{\tau}$ lies in $\tilde{\tau}$
iii. The intersection of any finite members $\tilde{\tau}$ also lies in $\tilde{\tau}$
Then $\tilde{\tau}$ is called fuzzy digital topology determined by d.
Proof:
i. Given any $\tilde{P}_i \in X \times Y \times I$ and $r > 0$. Then by Definition 5.2, $(\tilde{P}_i, r) \subseteq X \times Y \times I$ is fuzzy open.$\emptyset$ is fuzzy open in any set.
ii. Let $\tilde{S} = \bigcap_i \tilde{S}_i$ is an element in $\tilde{S}$. Then from definition of union of set, $\tilde{P}_i \in \tilde{S}_i$ for $i \in I$. Because of $\tilde{S}_i$ is fuzzy open, then there exist $r > 0$ such that $S(\tilde{P}_i, r) \subseteq \tilde{S}_i \subseteq \tilde{S}$.Since $\tilde{P}_i$ is arbitrary, it follow that $\tilde{S}$ is fuzzy open in $X \times Y \times I$.
iii. Let $\tilde{S}, 1 \leq i \leq n$, be a finite collection of fuzzy point relation and fuzzy open in $X \times Y \times I$. Now show that their intersection $\bigcap_i^n \tilde{S}_i$ is fuzzy open.If $\bigcap_i^n \tilde{S}_i = \emptyset$, then the intersection is fuzzy open. If not, let $\tilde{P}_i \in \bigcap_i^n \tilde{S}_i$ be arbitrary. Since $\tilde{S}$ is fuzzy open and $\tilde{P}_i \in \tilde{S}_i$, there exist $r_i > 0$ such that $S(\tilde{P}_i, r) \subseteq \tilde{S}_i$. Let

$r = \min\{r_i : 1 \le i \le n\}$. Then clearly, $S(\tilde{P}_i, r) \subseteq S(\tilde{P}_i, r_i) \subseteq \tilde{S}_i$, for each i, that is $S(\tilde{P}_i, r) \subseteq \bigcap_{i}^{n} \tilde{S}_i$.

6. Fuzzy Cubic Bezier Curve Approximation in FTDS

In Wahab and Zulkifly (2017) has introduced FCPR from fuzzy point relation to build a new fuzzy Bezier curve modeling. In this paper, we extend the result from Wahab and Zulkifly (2017) to build fuzzy Bezier curve in FTDS and focus on cubic case where $n = 3$.

Definition 6.1. *Wahab and Zulkifly (2017) Let $\tilde{C}_{P_R i} = \{C_{P_R 0}, \tilde{C}_{P_R 1}, ..., \tilde{C}_{P_R n}\}$ are fuzzy control point relation (FCPR) where it is a set $n+1$ points that shows a positions and coordinates of a location and also control the shape of a curve. The $\tilde{C}_{P_R i}$ also can be write as $\{(P_i, z_i)_0, (P_i, z_i)_1, ..., (P_i, z_i)_n\}$*

Below we show that FCPR satisfies fuzzy digital topologyâs condition.

Theorem 6.1. *Let $(X \times Y \times I, \tilde{\tau})$ be FTDS and $\tilde{S} \in \tau$ is fuzzy point relation. If $\tilde{C}_{P_R i} \subseteq \tilde{S}$ denote as FCPR, then fuzzy digital topology on $X \times Y \times I$.*

Proof:

i. $\emptyset, \tilde{C}_{P_R i} \subseteq \tilde{S}$. Therefore $\emptyset, \tilde{C}_{P_R i} \in \tilde{\tau}$

ii. Let $\tilde{C}_{P_R i}, \tilde{C}_{P_R j} \in \tilde{\tau}$. If $\tilde{C}_{P_R i} \subseteq \tilde{S}$ and $\tilde{C}_{P_R j} \subseteq \tilde{S}$, then $\tilde{C}_{P_R i} \cap \tilde{C}_{P_R j} \subseteq \tilde{S}$. Therefore $\tilde{C}_{P_R i} \cap \tilde{C}_{P_R j} \in \tilde{\tau}$.

iii.Let $\{\tilde{C}_{P_R i} : i \in I\}$ subcollection for $\tilde{\tau}$. Because $\tilde{C}_{P_R i} \subseteq \tilde{S}$ for every $i \in I$, then $\bigcup_{i \in I} \tilde{C}_{P_R i} \subseteq \tilde{S}$. Therefore $\bigcup_{i \in I} \tilde{C}_{P_R i}$ is subset of $\tilde{S}$ and belong to $\tilde{\tau}$.

In Wahab and Zulkifly (2017) shows how the FCPR blend with Bernstein polynomial.

Definition 6.2. *Wahab and Zulkifly (2017) Let $\tilde{C}_{P_R i}$ be a FCPR and $B(t)$ be a Bezier curve with parameter t hence by blending it, fuzzy Bezier curve is defined as follow.*

$$\tilde{B}(t) = \sum_{i=0}^{n} \tilde{C}_{P_R i}, B_{n,i}(t), 0 \le t \le 1 \tag{4}$$

with $B_{n,i}(t) = \begin{pmatrix} n \\ i \end{pmatrix} t^i (1-t)^{n-i}$ known as Bernstein polynomial or blending function where $\begin{pmatrix} n \\ i \end{pmatrix} = \frac{n!}{i!(n-i)!}$ are the binomial coefficients.

For degree of $n = 3$ (cubic case), fuzzy cubic Bezier curve also can be written as $\tilde{B}(t) = \sum_{i=0}^{3} \tilde{C}_{P_R i}, B_{n,i}(t), 0 \leq t \leq 1$ or

$$\tilde{B}(t) = \tilde{C}_{P_R 0}, B_{n,0} + \tilde{C}_{P_R 1}, B_{n,1} + \tilde{C}_{P_R 2}, B_{n,2} + \tilde{C}_{P_R 3}, B_{n,3} \quad (5)$$

where it consist of four FCPR.

By using previous section (section 4 and 5), now we show how to generate the visualization of fuzzy cubic Bezier curve approximation in example 6.1 below.

Example 6.1. *To construct a fuzzy cubic Bezier curve, let $n = 3$ and $X \times Y \times I \neq \emptyset$ be a fuzzy digital space. Therefore, FCPR $\tilde{C}_{P_R 3} = \{(P_i, z_i)_0, (P_i, z_i)_1, (P_i, z_i)_2, (P_i, z_i)_3, \} \in \tilde{S}$ where $P_i = (x_i, y_i)$ and $z_i \in (0, 1)$. If $\tilde{\tau}$ is a collection of fuzzy point relation and $\tilde{C}_{PRi} \in \tilde{S}$, then $(X \times Y \times I, \tilde{\tau})$ is FTDS where it satisfies three conditions above (see definition 4.1). Now let $d(\tilde{C}_{P_R 0}, \tilde{C}_{P_R 1})$ is the distance between $\tilde{C}_{P_R 0}$ and $\tilde{C}_{P_R 1}$ Then d is fuzzy digital metric where satisfies some properties above (see definition 5.1). For Bernstein polynomial, $B_{3,i}$ is as follows*

$$B_{3,0} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} t^0(1-t)^{3-0} = (1-t)$$
$$B_{3,1} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} t^1(1-t)^{3-1} = 3t(1-t)^2$$
$$B_{3,2} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} t^2(1-t)^{3-2} = 3t^2(1-t)$$
$$B_{3,3} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} t^3(1-t)^{3-3} = t^3$$

Therefore, fuzzy cubic Bezier curve is

$$\tilde{B}(t) = \sum_{i=0}^{3} \tilde{C}_{P_R i}, B_{n,i}(t), 0 \leq t \leq 1$$
$$= (1-t)^3\tilde{C}_{P_R 0} + (3t(1-t)^2)\tilde{C}_{P_R 1} + (3t^2(1-t))\tilde{C}_{P_R 2} + (t^3)\tilde{C}_{P_R 3}$$
$$= (1-t)^3(P_i, z_i)_0 + (3t(1-t)^2)(P_i, z_i)_1 + (3t^2(1-t))(P_i, z_i)_2 + (t^3)(P_i, z_i)_3$$
$$= [(1-t)^3, 3t(1-t)^2, 3t^2(1-t), t^3][(P_i, z_i)_0, (P_i, z_i)_1, (P_i, z_i)_2, (P_i, z_i)_3]^\intercal$$
$$= [t^3, t^2, t, 1] \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} (P_i, z_i)_0 \\ (P_i, z_i)_1 \\ (P_i, z_i)_2 \\ (P_i, z_i)_3 \end{bmatrix}$$

Below we show some visualization of fuzzy cubic Bezier curve in FTDS. The triangle form in figure 4 is to shows that there exist a distance, d between every FCPR in FTDS where d satisfies the fuzzy metric's properties (refer 5.1).

Below is an algorithm to generate the fuzzy Bezier curve in FTDS. Given fuzzy digital space and some fuzzy point relation, $S \subseteq X \times Y \times I$. This paper

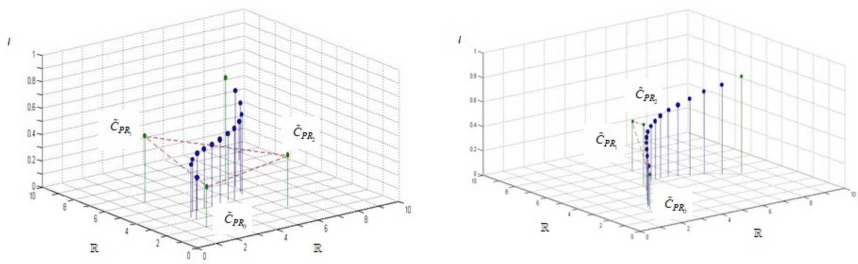

Figure 4: Example of fuzzy cubic Bezier curve in $\mathrm{R}\times\mathbb{R}\times\mathbb{I}$

considering cubic case $n = 3$.
Algorithm 1:
Step 1: Define $n+1$ FCPR in fuzzy point relation, $\tilde{C}_{P_R 3} = \{\tilde{C}_{P_R 0}, \tilde{C}_{P_R 1}, \tilde{C}_{P_R 2}, \tilde{C}_{P_R 3}\}$ where $\tilde{C}_{P_R i} \in \tilde{S}$ need to satisfy fuzzy digital topology properties.
Step 2: Measure the distance, d where $\tilde{C}_{P_R 3} = \{\tilde{C}_{P_R 0}, \tilde{C}_{P_R 1}, \tilde{C}_{P_R 2}, \tilde{C}_{P_R 3}\}$ where d satisfies the fuzzy digital metric properties.
Step 3:Calculate the Bernstein polynomial $B_{3,i} = \begin{pmatrix} 3 \\ i \end{pmatrix} t^i(1-t)^{3-i}$.
Step 4:Blend the FCPR with Bernstein polynomial
$\tilde{B}(t) = \sum_{i=0}^{3} \tilde{C}_{P_R i}, B_{3,i}(t), 0 \leq t \leq 1.$
Step 5: Write the solution in matrix form $\tilde{B}(t) = [B_{3,i}][\tilde{C}_{P_R i}]^{\intercal}$.
Step 6: Collect the coefficients of parameter and rewrite the step 5 as

$$\tilde{B}(t) = [t^3, t^2, t, 1]N \begin{bmatrix} (P_i, z_i)_0 \\ (P_i, z_i)_1 \\ (P_i, z_i)_2 \\ (P_i, z_i)_3 \end{bmatrix} \text{ where } N = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

7. Conclusions

In this paper, we improves the definition of FTDS by introduced fuzzy digital space concept and use fuzzy point relation as element in fuzzy digital topology that satisfies three conditions. Then, we introduce a concept of fuzzy metric digital space where there exist some distance between the fuzzy point relation in FTDS. Then we proceed to main purpose which is to build fuzzy cubic Bezier curve in FTDS. The main point to construct fuzzy cubic Bezier curve is to have FCPR. FCPR in fuzzy digital topology is then blended with the Bernstein polynomial to generate fuzzy cubic Bezier curve in FTDS. This work can be expanding to fuzzy cubic Bezier surface and to build another curve and surface for B-Spline.

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